# Application of Different Kinds of Approximation in Signals Restoration


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***Abstract* – It is proposed the method of continuous signals restoration by different kinds of approximation: Lagrange polynomial, Kotelnikov interpolation series, linear and cubic splines, Haar wavelet and Kotelnikov-Shannon wavelet. The minimum value of mean-square deviation is used as a criterion. The program witch allow to define optimal method of different class of signals restoration is composed.**



***Keywords: restoration; signal; approximation; Kotelnikov series; spline functions; wavelet functions; cubic spline; deviation; software product; mean-square deviation.***


## I. Introduction

Significant attention is paid to the signals restoration in telecommunications. It is connected with the fact that the substantial part of analog signals requires the restoration at different stages of analog-to-digital conversion. Furthermore, to use a single approach to signals restoration is not always possible. Indeed, original signals can differ in forms, make hops or “splashes” on the amplitude. The traditional method of signal restoration is the application of Kotelnikov series. However, the application of Kotelnikov series as the approximation function is effective in the infinite interval of time, while in the real systems it is necessary to examine a signal in a certain limited interval. In this case, there appears an error in signals restoration by means of Kotelnikov series. Therefore, to examine other kinds of approximation is reasonable. The following Lagrange exponential polynomial, spline function, wavelet function and many others are examined as the approximation functions [1-4]. In certain cases [1-3] the application of spline functions and wavelet functions allows to reach the best results of continuous signal restoration in the finite time interval, rather than with the application of Kotelnikov series.

The signals restoration by means of Kotelnikov interpolation series, cubic spline and wavelet functions were examined by the authors in works [1-4]. In works [1-3] the signals of linear frequency modulation, the random signals, as well as the signals, which are characterized by the amplitude steepness were investigated. Restoring such signals by Kotelnikov series and cubic splines, the estimations of restoration error are obtained and it is shown that under specific conditions the application of cubic splines is more reasonable. In work [4] it is found that for some signals the application of wavelet functions as approximation ones is more preferential in comparison with the use of Kotelnikov series and cubic splines. The task to select the optimum method of signals restoration, depending on their original kind, emerges.

In this work let us consider the restoration of different signal kinds by means of the above mentioned methods, that will allow us to give recommendation on the application of one or another restoration method depending on the examined signal.

The objective of this work is the comparison of the signals restoration methods with the use of different approximation functions for their more precise restoration.

To achieve the set objective let us find errors of original signal restoration by means of different approximation functions. As the criterion we will use a minimum value of mean-square deviation between the original signal and the approximation function. We will consider Kotelnikov interpolation series, Lagrange polynomial, spline function and wavelet function as the approximation functions.

## II. The Application of Kotelnikov Series and Spline Functions in Continuous Signals Restoration

Let us consider the continuous signal $f(x)$ in the interval $0 \le x \le 1$ that is given by the function of the form [2, 3]:

$$f(x) = \frac{\sin(16\pi(x-\frac{1}{2})}{8\pi(x-\frac{1}{2})} - \frac{3\sin(8\pi(x-\frac{1}{2})}{8\pi(x-\frac{1}{2})}, \quad (1)$$

It is considered that $f(x)$ is limited on the spectrum $F_{\max} = 8$ kHz, then the sample spacing $\tau = \frac{1}{2F_{\max}}$.

Let us consider samples of the original signal $f(x)$ as interpolation points. Let be given function values as $f_k = f(x_k)$, $k = \overline{0, N-1}$, in the interval [0;1] in mesh points $\Delta$: $0 = x_0 < x_1 < ... < x_N = 1$. The given original signal $f(x)$ belongs to the class of functions for which

$|f(x)| < \frac{A}{|x^2|}$, where $f(x)$ is original signal, $A$ is a certain constant. To restore the signals of such class, Kotelnikov interpolation series and cubic spline were examined in works [2, 3].

Fig.1 illustrates the obtained results of given above signals of restoration, examined in the works [2, 3], particularly: diagram of the given signal $f(x)$ (diagram 1), values of function $f(x)$ in the mesh points (diagram 2), the diagram of the restored signal by Kotelnikov series (diagram 3), the diagram of the restored signal by the cubic spline (diagram 4).

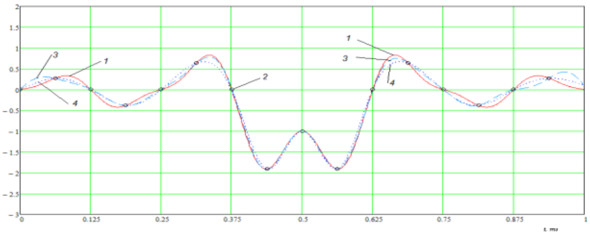

Fig. 1 – Original continuous signal $f(x)$ and its restoration by Kotelnikov series and cubic spline.

The obtained results in [2, 3] show that in the intervals of interpolation "with amplitude smooth change", where the form of the signal continuously changes without "amplitude steepness" the best results are given by spline approximation, and in the intervals with "amplitude steepness" – by Kotelnikov interpolation series. As it is shown in [2], the approximation error by application of Kotelnikov series and cubic spline can reach 6,2% and 9,2% in the first and the second cases respectively due to the fact, that with the uniform sampling the sample points can not coincide with the amplitude maximums or the minimums. It is found that the use of cubic spline as the approximation function during continuous signals restoration is reasonable for the signals, which do not have bumping parameters changes. At the same time, it is much more difficult to select the approximation function for the signals, which possess the amplitude hops and "rapid" oscillations.

Therefore it would be reasonable to consider another restoration method, for which this approximation error would be less. In our opinion, such method is wavelet transform device.

## III. The Application of Wavelet Functions of Continuous Signals Restoration

Recently special development was given to the wavelet analysis device in information theory, theory of coding, theory of signals and images.

According to [5], any function $f(x)$ from Hilbertia space $L^2(R)$ can be expressed in the series of the form:

$$f(x) = \sum_k s_{j_n,k}\varphi_{j_n,k}(x) + \sum_{j\ge j_n,k} d_{j,k}\psi_{j,k}(x) \qquad (2)$$

where $f(x)$ – temporary function of the original signal, $\varphi_{j,k}(x)$ – scaling function (father wavelet), $\psi_{j,k}(x)$ – basic wavelet function (mother wavelet), $s_{j,k}$ and $d_{j,k}$ – wavelet coefficients, $j_n$ – scaling level, $k$, $j$, $j_n$ – natural numbers.

Similarly to [5], wavelet coefficients $s_{j,k}$ and $d_{j,k}$ are calculated according to formulas:

$$s_{j-1,k} = \sum_m h_m s_{j,2k+m}\,,\; d_{j-1,k} = \sum_m g_m s_{j,2k+m}\,, \qquad (3)$$

where $h_m$, $g_m$ – approximation and expansion coefficients.

Scaling function $\varphi_{j,k}(x)$ and basic wavelet function $\psi_{j,k}(x)$ are obtained from the basic wavelet with the scaled transformation $1/2^k$ and shifts $j/2^k$ [5]:

$$\psi_{j,k} = 2^{j/2}\psi(2^j x - k)\,,\; \varphi_{j,k} = 2^{j/2}\varphi(2^j x - k)\,, \qquad (4)$$

where $\psi(x) = \sqrt{2}\sum_k g_k \varphi(2x-k)$ – mother wavelet, $\varphi(x) = \sqrt{2}\sum_k h_k\varphi(2x-k)$ – father wavelet,

$g_k = (-1)^k h_{(N-1)-k-1}$, $h_k = \sqrt{2}\int \varphi(x)\overline{\varphi}(2x-k)dx$ – expansion coefficient, $k, N$ – natural numbers.

In the case of signal restoration on the most detailed level, i.e. $j = j_{\max}$, coefficients $s_{j_{\max}}$ are calculated as follows [5]:

$$s_{j\max,n} = f(n\tau)/2^{\frac{j\max}{2}}\,, \qquad (5)$$

where $\tau$ - sampling interval.

Let us consider the restoration of the continuous signal $f(x)$ of the form (1) by wavelet function. To restore the signal let us use Kotelnikov-Shannon wavelet for which basic wavelet function $\psi(x)$ and scaling function $\varphi(x)$ take the following form [6]:

$$\psi(x) = 2\frac{\sin 2\pi(x-1/2) - \sin \pi(x-1/2)}{\pi(x-1/2)}\,, \qquad (6)$$

$$\varphi(x) = \begin{cases} \sin \pi x/\pi x, & x \ne 0, \\ 1, & x = 0, \end{cases}$$

Mother wavelet $\psi(x)$ and scaling function $\varphi(x)$ are connected by the relations (4), besides, for Kotelnikov-Shannon wavelet [6]:

$$g_k = (-1)^k h_{N-1-k}\,,\; k = \overline{0, N-1}$$

$$h_k = \begin{cases} 1/\sqrt{2}, & k = 0, \\ (-1)^{(k-1)/2}\sqrt{2}/\pi k, & k - \text{odd}, \\ 0, \text{ for the rest } k \end{cases} \qquad (7)$$

where $N$ – natural number, which defines the coefficients $h_k$ and $g_k$ number.

The necessary condition of the signal restoration by the wavelet function is its definition by samples number that is equal to $N = 2^j$, where $j \ge 1$ defines the maximum possible number of scaling levels [5]. In our case $j = j_{\max} = 4$, $N = 2^4 = 16$.

Lets us calculate the $h_k$ and $g_k$ expansion by the formula (7). The obtained values are given in the table 1.

TABLE 1. VALUES OF $h_k$ AND $g_k$ COEFFICIENTS

| **k** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** | **13** | **14** | **15** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $h_k$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{2}}{\pi}$ | 0 | $-\frac{\sqrt{2}}{3\pi}$ | 0 | $\frac{\sqrt{2}}{5\pi}$ | 0 | $-\frac{\sqrt{2}}{7\pi}$ | 0 | $\frac{\sqrt{2}}{9\pi}$ | 0 | $-\frac{\sqrt{2}}{11\pi}$ | 0 | $\frac{\sqrt{2}}{13\pi}$ | 0 | $-\frac{\sqrt{2}}{15\pi}$ |
| $g_k$ | $h_{15}$ | 0 | $h_{13}$ | 0 | $h_{11}$ | 0 | $h_9$ | 0 | $h_7$ | 0 | $h_5$ | 0 | $h_3$ | 0 | $h_1$ | $-h_0$ |

Table 2 shows the number values of $s_{j,k}$ and $d_{j,k}$ coefficients, obtained according to (3), besides, $s_{0,k}$ coefficients are values of function $f_k = f(x_k)$ in the interpolation points.

Lets us perform $f(x)$ signal restoration by the formula (2) [5], using Kotelnikov-Shannon wavelet, for which $\psi(x)$ and $\varphi(x)$ are defined by the formula (6) [6].

TABLE II. VALUES OF $S_{J,K}$ AND $d_{J,K}$ COEFFICIENTS

| | $k$ | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** |
|---|---|---|---|---|---|---|---|---|---|
| ***j=1*** | $s_{j,k}$ | 0,217 | 0 | 0 | -0,478 | -1,714 | 0 | 0 | 0,123 |
| | $d_{j,k}$ | -0,048 | 0,068 | -0,113 | 0,338 | 0,161 | -0,113 | 0,068 | -0,048 |

| | $k$ | **0** | **1** | **2** | **3** |
|---|---|---|---|---|---|
| ***j=2*** | $s_{j,k}$ | 0,134 | -0,026 | -1,065 | -0,095 |
| | $d_{j,k}$ | 0,082 | 0,318 | -0,443 | -0,082 |

| | $k$ | **0** | **1** |
|---|---|---|---|
| ***j=3*** | $s_{j,k}$ | 0,097 | -0,796 |
| | $d_{j,k}$ | 0,103 | -0,191 |

| | $k$ | **0** |
|---|---|---|
| ***j=4*** | $s_{j,k}$ | -0,29 |
| | $d_{j,k}$ | 0,062 |

According to the above mode calculations the original signal can be restored by Kotelnikov-Shannon wavelet. Fig. 2 illustrates the signal restoration of the form (1) in the interval $[0;1]$ by means of different approximation functions, and Fig. 3 – in the interval $[0{,}625;0{,}75]$: the diagram of the given signal $f(x)$ (diagram 1), the approximation of the original signal $f(x)$ by Kotelnikov series (diagram 2), the approximation of the original signal $f(x)$ by cubic spline (diagram 3), the approximation of the original signal $f(x)$ by Kotelnikov-Shannon wavelet (diagram 4).

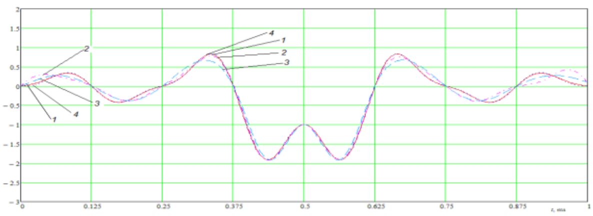

Fig. 2 Restoration of the original signal $f(x)$ by Kotelnikov series, cubic spline and Kotelnikov-Shannon wavelet in the interval $[0;1]$.

## IV. COMPARISON OF THE RESULTS OF CONTINUOUS SIGNALS RESTORATION

To compare the results of the original $f(x)$ signal restoration by Kotelnikov-Shannon wavelet let us consider the results obtained earlier in works [2, 3], the approximation of the same signal by Kotelnikov series and cubic spline. As we can see from the Fig. 2 and 3, the proposed method of continuous signals restoration, which are characterized by the amplitude abrupt changes, based on wavelet functions allows to obtain the best results of restoration (curve 4), rather than restoration methods by Kotelnikov series (curve 2) and by cubic spline (curve 3).

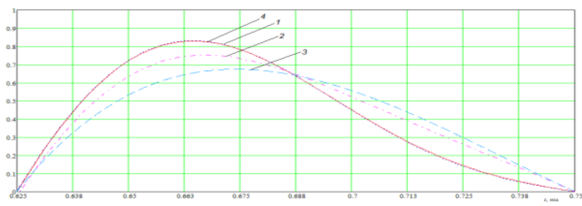

Fig. 3 Restoration of the original signal $f(x)$ by Kotelnikov series, cubic spline and Kotelnikov-Shannon wavelet in the interval $[0{,}625;0{,}75]$.

To estimate the fact how precisely one or another method (Kotelnikov series, cubic spline or Kotelnikov-Shannon wavelet) restores the original signal, let us define the value of deviation $\varepsilon_n$ of the restored signal values from the original signal. The results are given on the Fig. 4.

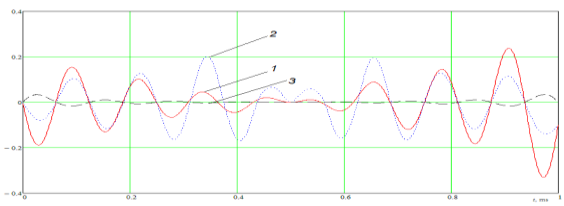

Fig. 4 The value of $\varepsilon_n$ deviation of restored signal values from its original value

Fig. 4 illustrates: diagram $\varepsilon_n$ with the approximation of the original signal $f(x)$ by Kotelnikov series (curve 1), diagram $\varepsilon_n$ with the approximation of the original signal $f(x)$ by cubic spline (curve 2), diagram $\varepsilon_n$ with the approximation of the original signal $f(x)$ by Kotelnikov-Shannon wavelet (curve 3).

Similarly to [2], let us define the mean-square deviation (MSD) of the restored and original signals difference. The results of the MSD values of the restored and original signals are given in table 3.

TABLE III. DEVIATION VALUES $\varepsilon_n$ AND MSD $\sigma_n$ OF THE RESTORED SIGNAL FROM ORIGINAL ONE BY APPROXIMATION FUNCTIONS

| № | Interval | Number limits of the interval | $\varepsilon_n$ Kotelnikov series | $\varepsilon_n$ cubic spline | $\varepsilon_n$ Kotelnikov-Shannon wavelet | $\sigma_n$ Kotelnikov series | $\sigma_n$ cubic spline | $\sigma_n$ Kotelnikov-Shannon wavelet |
|---|---|---|---|---|---|---|---|---|
| 1 | $[t_0, t_1]$ | [0; 0,063] | 0,119 | 0,05 | 0,021 | 0,059 | 0,025 | 0,01 |
| 2 | $[t_1, t_2]$ | [0,063; 0,125] | 0,098 | 0,066 | -0,011 | 0,048 | 0,032 | 0,0051 |
| 3 | $[t_2, t_3]$ | [0,125; 0,188] | 0,083 | 0,074 | 0,0065 | 0,041 | 0,036 | 0,0032 |
| 4 | $[t_3, t_4]$ | [0,188; 0,25] | 0,066 | 0,082 | 0,0044 | 0,031 | 0,038 | 0,0021 |
| 5 | $[t_4, t_5]$ | [0,25; 0,313] | 0,039 | 0,103 | 0,0029 | 0,023 | 0,051 | 0,0014 |
| 6 | $[t_5, t_6]$ | [0,313; 0,375] | 0,023 | 0,128 | 0,0019 | 0,02 | 0,061 | 0,0092 |
| 7 | $[t_6, t_7]$ | [0,375; 0,438] | 0,027 | 0,106 | 0,0011 | 0,017 | 0,053 | 0,00055 |
| 8 | $[t_7, t_8]$ | [0,438; 0,5] | 0,012 | 0,041 | 0,00035 | 0,0072 | 0,022 | 0,00022 |
| 9 | $[t_8, t_9]$ | [0,5; 0,563] | 0,0048 | 0,036 | 0,00035 | 0,0049 | 0,02 | 0,00022 |
| 10 | $[t_9, t_{10}]$ | [0,563; 0,625] | 0,022 | 0,099 | 0,0011 | 0,015 | 0,049 | 0,00053 |
| 11 | $[t_{10}, t_{11}]$ | [0,625; 0,688] | 0,059 | 0,121 | 0,0018 | 0,026 | 0,062 | 0,00095 |
| 12 | $[t_{11}, t_{12}]$ | [0,68; 0,75] | 0,079 | 0,107 | 0,0029 | 0,035 | 0,05 | 0,0014 |
| 13 | $[t_{12}, t_{13}]$ | [0,75; 0,813] | 0,091 | 0,081 | 0,0043 | 0,046 | 0,04 | 0,0021 |
| 14 | $[t_{13}, t_{14}]$ | [0,813; 0,875] | 0,117 | 0,076 | 0,0012 | 0,056 | 0,036 | 0,0031 |
| 15 | $[t_{14}, t_{15}]$ | [0,875; 0,938] | 0,149 | 0,072 | -0,01 | 0,075 | 0,037 | 0,0054 |
| 16 | $[t_{15}, t_{16}]$ | [0,938; 1] | 0,225 | 0,102 | 0,02 | 0,092 | 0,038 | 0,0065 |

To compare the methods of signals restoration by means of different approximation functions (Kotelnikov series, Lagrange polynomial interpolation, linear spline functions, cubic spline functions, as well as wavelet functions) the software realization “*Signal Restoration*”, which realizes the mentioned methods of the signals restoration is developed [4]. The given product allows to perform the signals restoration of different classes (both continuous and discrete) by all indicated methods, as well as to obtain the estimation of restoration error according to the MSD criterion of the restored signal from the original one. The best method of signal restoration based on the obtained MSD is chosen.

## CONCLUSIONS

1. As a result of carried out analysis of the continuous signals restoration on their samples by Kotelnikov series, cubic spline and Kotelnikov-Shannon wavelet it is shown, that the first two restoration methods do not consider such special features of signal as the amplitude abrupt change, which leads to the restored signal values deviation from its original value. The use of Kotelnikov-Shannon wavelet allows to decrease this deviation. This implies, that the mathematical device of wavelet analysis can be used for the signals restoration, which are characterized by the presence of rapid oscillations.

2. The estimations of the restoration error of different signal classes by the functions, pointed out above, are obtained, which allowed speaking about the advantage of a certain method of the signals restoration depending on its kind, i.e., the class of signals.

3. The program, which allows to find the restoration errors of different signals based on the MSD minimum value criterion by means of Lagrange polynomial, Kotelnikov interpolation series, linear and cubic splines, Haar wavelet and Kotelnikov-Shannon wavelet is developed, which allow to give recommendations regarding the selection of the approximation function based on the mean-square functions deviation criterion for different signal kinds.

## REFERENCES


[1] Strelkovskaya I.V. Continuous signals restoration based on Kotelnikov series and cubic splines / I.V.Strelkovskaya, D.Y.Bukhan // Radioelectronics: All-Ukr. Interdep. science-research coll. – 2007. – Iteration. 151. – P. 181-185.

[2] Strelkovskaya I.V. Comparative analysis of continuous signals restoration by Kotelnikov series and spline-functions / I.V.Strelkovskaya, O.V.Lysiuk, R.V.Zolotukhin // Eastern-European Journal of Enterprise Technology. – 2013. – №2/9(62). – P. 12-15.

[3] Strelkovskaya I.V. Investigation of continuous signals restoration methods / I.V.Strelkovskaya, O.V.Lysiuk, R.V.Zolotukhin // The VII International Scientific Research Conference “Telecommunications Problems”: Conference Proceedings., April 16-19, 2013,– K.NTUU”KPI” Kyiv, 2013. – P. 82–84.

[4] I. Strelkovskaya Software development of the different signal restoration methods / I. Strelkovskaya, O. Lysiuk, R. Zolotukhin // The VIII International Scientific Research Conference “Telecommunications Problems”:Conference Proceedings., April 22-25, 2014,–K.NTUU”KPI” Kyiv, 2014. – P. 128–130.

[5] Dremin I.M. Wavelets and their application / I.M.Dremin, O.V.Ivanov, V.A.Nechitailo // Progress of physic sciences. – 2001. T.171. – №5. – p. 465-501.

[6] Smolentsev N.K. Fundamentals of wavelets theory. Wavelets in MATLAB / N.K.Smolentsev – M.: DMK Press, 2005. – 304.